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Étale descent for real number fields

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Abstract

In this paper we verify the strong Quillen–Lichtenbaum conjecture for integers in real number fields at the prime two. That is, we prove that the Dwyer–Friedlander map from mod 2 algebraic K-theory to mod 2 étale topological K-theory is a weak equivalence on zero-connected covers for two integers in real number fields. The proof is given by comparing two explicit calculations. © 2002 Published by Elsevier Science Ltd.

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1. Introduction

The Quillen–Lichtenbaum conjectures in algebraic K-theory are important conjectures with a rich history. One version of the conjectures predicts that the algebraic K-theory spectrum and the étale topological K-theory spectrum of a number field have homotopy equivalent zero-connected covers. The latter spectrum is defined by means of more algebro-geometric methods than the algebraic K-theory spectrum, and depends upon the étale homotopy type of the number field. Étale cohomology may be used to compute étale topological K-theory, in the same way as singular cohomology can be used to compute complex topological K-theory. From this point of view, the philosophy behind the Quillen–Lichtenbaum conjectures is that the algebraic K-spectrum of a number field depends only upon the étale homotopy type of the number field, and the algebraic K-groups are just some étale cohomology groups in disguise. These conjectures have had an immense impact on the field of algebraic K-theory over the last 25 years. It seems therefore appropriate to recapitulate why these conjectures are so popular and also indicate why they are important in some other related fields of mathematics.

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The first versions of the conjectures called the Quillen–Lichtenbaum conjectures today were suggested by Lichtenbaum in [24] and by Quillen in [34]. Lichtenbaum [24] proposed some striking conjectures of a strong arithmetic flavor. Conjecture 2.4 in [24] predicts that the algebraic K-groups of a totally real number field F determines the values of the zeta-function of F at odd negative integers. This has quite recently been verified at the prime two in [36]. Lichtenbaum also posed the question of when the covolume of the Borel regulator of a number field F could be expressed in terms of the algebraic K-groups of F and the leading coefficient in the Taylor expansion of the zeta-function of F at negative integers. This conjecture may be viewed as a generalization of the analytic class number formula in number theory. See [21] for results at odd primes.

Let A be a finitely generated regular ring and ℓ a rational prime number. Then Quillen [34] conjectured that there exists an Atiyah–Hirzebruch type spectral sequence:

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}\left(A\left[\frac{1}{\ell}\right]; \mathbf{Z}_\ell\left(\frac{q}{2}\right)\right) \Rightarrow \mathrm{K}_{p-q}\left(A\left[\frac{1}{\ell}\right]\right) \otimes \mathbf{Z}_\ell.$$

The coefficient sheaves are to be interpreted as zero for q odd. If A is the ring of integers in a number field and ℓ is odd, then this spectral sequence would degenerate on the E_2-page and give an étale cohomological description of the ℓ-adic completion of the K-groups of A. Many papers on algebraic K-theory of number fields has been based upon this conjecture.

The next influential work was Soulé's treatise on étale Chern classes in [39]. He constructed Chern classes from algebraic K-theory to étale cohomology, and proved that these were surjective in many cases. This inspired Dwyer and Friedlander to introduce their étale topological K-theory, and further they could rephrase the Quillen–Lichtenbaum conjecture in terms of their étale theory. We give a brief review of this next.

Let F be a number field with ring of ℓ-integers R_F. The space $\mathrm{BGL}_n(R_F)$ can be identified with the basepoint component of the space of maps from $\mathrm{Spec}(R_F)$ to the bar construction of the affine group scheme $\mathrm{GL}_{n,\mathbf{Z}[1/\ell]}$ considered as schemes over $\mathrm{Spec}(\mathbf{Z}[1/\ell])$. We let $\mathrm{BGL}_n^{\text{ét}}(R_F)$ be the basepoint component of the space of maps from the étale homotopy type $\mathrm{Spec}(R_F)_{\text{ét}}$ of $\mathrm{Spec}(R_F)$ (see [2,15]) to the fibrewise ℓ-adic completion of $(\mathrm{BGL}_{n,\mathbf{Z}[1/\ell]})_{\text{ét}}$ over $\mathrm{Spec}(\mathbf{Z}[1/\ell])_{\text{ét}}$. There results a natural map:

$$\mathrm{BGL}_n(R_F) \to \mathrm{BGL}_n^{\text{ét}}(R_F).$$

Assume R_F has finite mod ℓ étale cohomological dimension. Then one defines the étale K-theory space of R_F by passing to the sequential colimit of the diagram

$$\cdots \to \mathrm{BGL}_n^{\text{ét}}(R_F) \to \mathrm{BGL}_{n+1}^{\text{ét}}(R_F) \to \cdots .$$

This construction may be promoted to the spectrum level to produce the ℓ-adic étale topological K-theory spectrum $\mathrm{K}^{\text{ét}}(R_F)_\ell$ of R_F. If $\ell=2$ and F is a real number field, so R_F has infinite mod 2 étale cohomological dimension, then Dwyer and Friedlander [11] define $\mathrm{K}^{\text{ét}}(R_F)_\ell$ as the homotopy fixed point spectrum of the ℓ-adic étale topological K-theory spectrum of R_E where $\mathrm{Spec}(R_E) \to \mathrm{Spec}(R_F)$ is a finite étale Galois covering and R_E has finite mod 2 étale cohomological dimension. See also Proposition 7.1 in [10]. The mod ℓ^v étale topological K-theory spectrum $\mathrm{K}^{\text{ét}}/\ell^v(R_F)$ of R_F is defined as $\mathrm{K}^{\text{ét}}(R_F)_\ell$ smashed with the mod ℓ^v Moore spectrum. Likewise we let $\mathrm{K}(R_F)_\ell$ (resp. $\mathrm{K}/\ell^v(R_F)$) denote the ℓ-adically completed (resp. mod ℓ^v) algebraic K-theory spectrum of R_F.

In this paper we will consider the Dwyer–Friedlander maps

$$\phi_\ell : \mathrm{K}(R_F)_\ell \to \mathrm{K}^{\text{ét}}(R_F)_\ell$$

and

$$\phi/\ell^v : \mathrm{K}/\ell^v(R_F) \to \mathrm{K}^{\text{ét}}/\ell^v(R_F),$$

induced by the natural map of classifying spaces above.

Assume for a moment that F has finite mod ℓ étale cohomological dimension. Note that this assumption excludes real number fields for $\ell = 2$. Then Dwyer and Friedlander constructed in Proposition 5.1 of [10] a natural, strongly convergent right half-plane spectral sequence:

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}\left(R_F; \mathbf{Z}_\ell\left(\frac{q}{2}\right)\right) \Rightarrow \mathrm{K}^{\text{ét}}_{p-q}(R_F)_\ell.$$

There is a similar spectral sequence with mod ℓ^v coefficients. Note the analogy of this non-conjectural spectral sequence and the conjecture of Quillen [34] for algebraic K-theory. The following version of the Quillen–Lichtenbaum conjecture is stated in [11].

The strong Quillen Lichtenbaum conjecture. *The Dwyer–Friedlander map ϕ_ℓ is a weak equivalence on zero-connected covers for all ℓ.*

Dwyer and Friedlander were the first to give some evidence for this conjecture. The following result, whose proof is based upon the spectral sequence above, can be found in Theorem 8.5 of [10].

Theorem 1 (Dwyer–Friedlander). *Let ℓ be a rational prime and $v \geqslant 1$. If $\ell = 2$, assume that $v \geqslant 2$ and that F contains a primitive fourth root of unity. Then the induced map*

$$\phi_*/\ell^v : \mathrm{K}_*/\ell^v(R_F) \to \mathrm{K}^{\text{ét}}_*/\ell^v(R_F)$$

on homotopy is surjective for $* \geqslant 1$.

Similar surjectivity results were proved in [12] (under the slogan "Algebraic K-theory eventually surjects onto topological K-theory"). Snaith introduced the so-called Bott periodic algebraic K-theory in [38]. One defines that theory by inverting a canonical Bott element β in algebraic K-theory. Thomason used this construction in his masterpiece [45].

Theorem 2 (Thomason). *Keep the same assumptions as in the previous theorem. Then there exists a strongly convergent spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}\left(R_F; \mathbf{Z}/\ell^v\left(\frac{q}{2}\right)\right) \Rightarrow \mathrm{K}_{p-q}/\ell^v(R_F)[\beta^{-1}].$$

Moreover, the Dwyer–Friedlander map induces a weak equivalence:

$$\mathrm{K}/\ell^v(R_F)[\beta^{-1}] \to \mathrm{K}^{\text{ét}}/\ell^v(R_F).$$

Thomason's theorem is a milestone in algebraic K-theory, and remains valid for quite general schemes. Dwyer and Mitchell used this result to compute complex topological K-theory of the algebraic K-theory spectrum of smooth curves over finite fields, of local fields and of number fields.

See the papers [13,14]. Their results are expressed in terms of Iwasawa theory, and shows in a fascinating way how the arithmetic in a number field determines the K-theory spectrum of the number field. Mitchell has strengthened the spectral sequence result of Thomason by identifying the abutment to something closer to algebraic K-theory, namely its harmonic localization. See [27–29] for details and many other interesting results too.

The assumption that F contains a primitive fourth root of unity if $\ell = 2$ is imposed to ensure finite mod 2 étale cohomological dimension, and is also needed for the construction of Bott elements. The theme in this paper is essentially to remove that hampering assumption, and to see what we can say about the Dwyer–Friedlander map for F at the prime two. For this we will employ the recent advances in motivic cohomology and K-theory. We give a brief review of this next.

Beilinson suggested almost 20 years ago that there should exist a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p\left(X; \mathbf{Z}\left(\frac{q}{2}\right)\right) \Rightarrow \mathrm{K}_{p-q}(X)$$

for a quite general scheme X. The terms on the E_2-page are the so-called motivic cohomology groups. The Suslin–Voevodsky motivic complexes satisfy so many of the properties required by Beilinson, that the cohomology of the motivic complexes of Suslin–Voevodsky, which live in a triangulated category of motives, is what one calls motivic cohomology today. Results of Suslin [42], Friedlander–Voevodsky [17] and Voevodsky [46] show that Bloch's higher Chow groups agree with motivic cohomology for smooth schemes over a field which admits resolution of singularities. In [5], Bloch and Lichtenbaum constructed a spectral sequence for X the spectrum of a field:

$$\mathrm{E}_2^{p,q} = \mathrm{CH}^{-q}(X, -p-q) \Rightarrow \mathrm{K}_{p-q}(X).$$

Suslin and Voevodsky [43] proved that two-adic motivic cohomology of X agrees with two-adic étale cohomology of X in a certain range. This uses Voevodsky's proof of the Milnor conjecture in [47]. The Bloch–Lichtenbaum spectral sequence has more recently been reinterpreted by Friedlander–Suslin in [16] and extended further by Levine in [23]. By using a version of the mentioned Bloch–Lichtenbaum spectral sequence with finite coefficients and the other results above, Rognes and Weibel calculated the mod 2 algebraic K-groups of a number field up to extensions in [36]. Moreover, the same authors proved the strong Quillen–Lichtenbaum conjecture for totally imaginary number fields in [37].

Theorem 3 (Rognes–Weibel). *The strong Quillen–Lichtenbaum conjecture is true for totally imaginary number fields at the prime two.*

A proof of the Bloch–Kato conjecture (one of the most respectable and far-reaching conjectures concerning Galois cohomology of fields) from [4] at an odd prime ℓ would imply the strong Quillen–Lichtenbaum conjecture for F at ℓ. However, this is not clear at all for real number fields at $\ell = 2$. The arguments in [37] do not apply in this case, essentially because real number fields have infinite mod 2 étale cohomological dimension at the prime two and nobody has ever computed the mod 2 étale K-groups of a general real number field. I do not know of any results in the direction of making a comparison of the Bloch–Lichtenbaum spectral sequence with the Dwyer–Friedlander spectral sequence. In our approach to the strong Quillen–Lichtenbaum conjecture for real number fields we will not invoke the Dwyer–Friedlander spectral sequence, but rather replace it with the homotopy fixed point spectral sequence.

Our approach to the strong Quillen–Lichtenbaum conjecture is more from a homotopy-theoretic point of view. Let G be a discrete group, X a naive G-spectrum and EG a free contractible G-space. By the G-homotopy fixed points of X we mean the function spectrum $X^{hG} = \mathrm{Map}_G(EG_+, X)$ defined in [25]. We may approximate the homotopy groups of X^{hG} by the conditionally convergent homotopy fixed point spectral sequence:

$$\mathrm{E}^2_{p,q} = \mathrm{H}^{-p}(G; \pi_q(X)) \Rightarrow \pi_{p+q}(X^{hG}).$$

This spectral sequence comes about from first filtering EG by skeleta and then applying the equivariant function spectrum functor. Equivalently we may filter X by an equivariant Postnikov tower. Greenlees and May have checked that the two constructions give the same spectral sequence, see [19].

From now on we assume that F is a real number field. Let $F \to E$ be a finite cyclic Galois extension with Galois group G where E is totally imaginary. If the extension is unramified outside the dyadic and infinite primes, then we get a Galois extension $R_F \to R_E$ of rings of two integers with Galois group G. Next, we consider the diagram of spectra

$$\begin{tikzcd} \mathrm{K}(R_F)_2 \arrow[r] \arrow[d] & \mathrm{K}(R_E)_2^{hG} \arrow[d] \\ \mathrm{K}^{\text{ét}}(R_F)_2 \arrow[r] & \mathrm{K}^{\text{ét}}(R_E)_2^{hG} \end{tikzcd}$$

The lower horizontal map is an equivalence by the construction of étale topological K-theory, and the right hand vertical map is an equivalence on zero-connected covers due to étale descent for totally imaginary number fields from [37]. This leads to a reformulation of the strong Quillen–Lichtenbaum conjecture for real number fields.

Étale descent for integers in real number fields at two. Let $F \to E$ be an extension as above. The strong Quillen–Lichtenbaum conjecture for a real number field F at the prime two is equivalent to the assertion that the map

$$\mathrm{K}(R_F)_2 \to \mathrm{K}(R_E)_2^{hG}$$

is a weak equivalence on zero-connected covers.

There is a canonical choice for E.

Notation. For the rest of the paper we let E be F extended with a primitive fourth root of unity. In this case $G = C_2$ is cyclic of order two.

We use the homotopy fixed point spectral sequence to compare the mod 2 homotopy groups of $\mathrm{K}(R_F)$ with the mod 2 homotopy groups of $\mathrm{K}(R_E)^{hC_2}$ via the natural map. Our main result verifies the strong Quillen–Lichtenbaum conjecture for real number fields at the prime two. Recall that a map of spectra induces a homotopy equivalence after two-adic completion if and only if the induced maps of mod 2 homotopy groups are isomorphisms.

Theorem 4. *The canonical map*

$$\mathrm{K}/2(R_F) \to \mathrm{K}/2(R_E)^{hC_2}$$

is a weak equivalence on zero-connected covers.

A recent paper of Mitchell [31] combined with our work gives a description of the two-adic homotopy type of the algebraic K-theory spectrum of R_F in terms of the Iwasawa-theory of the cyclotomic $\mathbf{Z}_2$-extension of F. See also [30].

It seems appropriate to say some words about the proof itself. There are several well-known irksome number-theoretic and homotopy-theoretic problems at the prime two. We will only mention a few of them in the introduction. The homotopy groups of $\mathrm{K}/2(R_E)$ are known from [36], see also [20].

Theorem 5 (Kahn, Rognes, Weibel). *There are isomorphisms*

$$\mathrm{K}_*/2(R_E) = \begin{cases} \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2) & \text{if } * \geqslant 1 \text{ odd}, \\ \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2) \oplus \mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2) & \text{if } * \geqslant 2 \text{ even} \end{cases}$$

The extension

$$0 \to \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2) \to \mathrm{K}_*/2(R_E) \to \mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2) \to 0$$

is split as a sequence of C_2-modules for $* \geqslant 2$ even.

Proof. The calculation of $\mathrm{K}_*/2(R_E)$ is Theorem 7.2 in [36]. The extension in question is split as an exact sequence of Abelian groups since E contains a primitive fourth root of unity. To get a splitting of C_2-modules we consider the diagram

$$\begin{tikzcd} \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2) \arrow[r] & \mathrm{K}_*/2(R_E) \arrow[r] & \mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2) \\ & \mathrm{K}_*/2(\mathbf{Z}[\tfrac{1}{2},i]) \arrow[u] \arrow[r, "\cong"] & \mathrm{H}^0_{\text{ét}}(\mathbf{Z}[\tfrac{1}{2},i];\mathbf{Z}/2) \arrow[u, "\cong"'] \end{tikzcd}$$

where the indicated isomorphisms are isomorphisms of C_2-modules. □

The first problem is to identify the groups on the E^2-page of the homotopy fixed point spectral sequence. That is, we must compute the cohomology groups $\mathrm{H}^{-p}(C_2;\mathrm{K}_*/2(R_E))$ where the generator of C_2 acts by complex conjugation on the groups $\mathrm{K}_*/2(R_E)$. This amounts to calculating the C_2 group cohomology of the étale cohomology groups of R_E. The obvious way to proceed now is to use the Lyndon–Hochschild–Serre spectral sequence. That seems to give only partial information. However, we compute all the groups by using the Lyndon–Hochschild–Serre spectral sequence for positive étale cohomology [9]. One advantage of positive étale cohomology is that the mod 2 positive étale cohomological dimension of a real number field is at most two. In this way, we somehow get around the problem with infinite mod 2 étale cohomological dimension. After some efforts we can write down explicitly what the groups on the E^2-page of the mod 2 homotopy fixed point spectral sequence for the extension $R_F \to R_E$ are. One way to detect permanent cycles is to consider the image of $\mathrm{K}/2(R_F)$ in $\mathrm{K}/2(R_E)$. All classes in that image are permanent cycles. Here we know the homotopy groups of $\mathrm{K}/2(R_F)$ up to extensions from [36]. Next, we employ work of Levine [23]. His extended Bloch–Lichtenbaum spectral sequence allows to compute explicitly what the image of $\mathrm{K}/2(R_F)$ in $\mathrm{K}/2(R_E)$ is. That calculation exhibits some infinite cycles, and leads to a calculation of the d^2-differentials from the odd rows. We compute the remaining d^2-differentials by comparing with the Lyndon–Hochschild–Serre spectral sequence.

The homotopy fixed point spectral sequence for the extension $\mathbf{R} \to \mathbf{C}$ becomes useful as we try to determine the d^3-differentials. The spectral sequence for the real numbers is in principle known from classical work of Atiyah [3]. Our proof uses results of Suslin in [40,41]. However, we record this in detail since we will need an explicit description of the differentials. We refrain from explaining the details in the computation of the d^3-differentials for the number field in this introduction, but the result is that the spectral sequence collapses at its E^4-page. There is a quite appealing homotopy-theoretic description of the classes that survive. The class η of the complex Hopf map $S^3 \to S^2$ generates the first stable stem, and its mod 4 reduction η_2 represents a non-trivial class in $\mathrm{K}_1/4(R_F)$. It turns out that all the classes in $\mathrm{K}_*/2(R_E)^{hC_2}$ can be written as one times or η_2 times or η_2^2 times a class in the image of $\mathrm{K}_*/2(R_F)$ in $\mathrm{K}_*/2(R_E)$. In fact, all the classes that survive on the first column are hit by classes from $\mathrm{K}_*/2(R_F)$ and multiplication by η_2 gives a surjection to the next two columns. There are no non-trivial classes in any of the other columns. An easy comparison of two ranks concludes the proof using [36].

We end this introduction with some rather speculative remarks. The strong Quillen–Lichtenbaum conjecture is discussed from a chromatic point of view in [48]. Let BP be the ℓ-local Brown–Peterson spectrum whose homotopy groups form a polynomial ring $\mathbf{Z}_{(\ell)}[v_1, \ldots, v_n, \ldots]$. The spectrum $\mathrm{BP}[v_n^{-1}]$ is defined as the telescope of the self map of BP given by multiplication by v_n. We let L_n be the Bousfield localization functor associated to $\mathrm{BP}[v_n^{-1}]$, see [8]. Waldhausen gives a reformulation of the strong Quillen–Lichtenbaum conjecture by asserting that $\mathrm{K}(R_F)_{(\ell)}$ should be L_1-local on zero-connected covers (for ℓ odd that is). We have proved that this is the case for $\ell = 2$. Non-linear algebraic K-theory is certainly not L_1-local, and it seems very interesting to study ring spectra that bridge the linear and non-linear worlds. We may also ask for a Quillen–Lichtenbaum conjecture for these.

2. The homotopy fixed point spectral sequence and the pairing

In this section we discuss in more detail the homotopy fixed point spectral sequence. Our perspective is based on [7,19]. We will sketch the construction of the spectral sequence, address the question of convergence, and then discuss the differentials. The differentials do not automatically satisfy a Leibniz rule if the spectrum in question does not have a product. A way to get around this problem has been anticipated in a series of papers by Rognes in [35], and we will apply the ideas in loc. cit. to get a pairing of spectral sequences for which there is a Leibniz rule. This part assumes only that we are dealing with ring spectra, and may very well be adopted to other theories than K-theory. In the following we will use the terminology from [6].

2.1. The HFP spectral sequence

Let X be a naive G-spectrum where G is a finite group. That is, each space in the spectrum is a pointed G-space and the structure maps are G-maps. The homotopy fixed point spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{H}^{-p}(G; \pi_q X) \Rightarrow \pi_{p+q}(X^{hG}) \tag{1}$$

is a homological half-plane spectral sequence with entering differentials. One can construct (1) from the increasing skeleton filtration $\{EG_+^{(n)}\}_{n\geqslant 0}$ of EG_+. The filtration is so that

$$EG^{(n,n-1)}:=EG_+^{(n)}/EG_+^{(n-1)} \cong G_+ \wedge \bigvee S^n,$$

where $\bigvee S^n$ is a finite wedge sum of n-spheres indexed over the elements in the n-fold product of G by itself. We write EG^n for the quotient of EG_+ by $EG_+^{(n)}$. By applying the functor $\mathrm{Map}_G(-,X)$ we get the tower

$$* \to \cdots \to \mathrm{Map}_G(EG^n,X) \to \mathrm{Map}_G(EG^{n-1},X) \to \cdots \to X^{hG}.$$

Next, we pass to homotopy groups, and the resulting exact couple

$$\begin{tikzcd}
\pi_* \mathrm{Map}_G(EG^n, X) \arrow[rr] & & \pi_* \mathrm{Map}_G(EG^{n-1}, X) \arrow[dl, "0"] \\
& \pi_* \mathrm{Map}_G(EG^{(n,n-1)}, X) \arrow[ul, "(-1)"] &
\end{tikzcd} \tag{2}$$

is the unrolled exact couple (0.1) in [6] for (1). From (2) one constructs (1) by using the standard procedure for exact couples. From (9.3) in [19] we have the displayed E^2-page of (1).

Let us denote by holim the homotopy inverse limit functor, and by hocolim the homotopy direct limit functor. The notion of a conditionally convergent spectral sequence is defined in [6].

Lemma 1. *The spectral sequence* (1) *is always conditionally convergent to the homotopy groups of* X^{hG}.

Proof. Consider the Milnor $\lim^1$ exact sequence

$$\begin{aligned}0 &\to \lim_n{}^1 \pi_{*+1} \mathrm{Map}_G(EG^n,X) \to \pi_* \operatorname{holim}_n \mathrm{Map}_G(EG^n,X)\\ &\to \lim_n \pi_* \mathrm{Map}_G(EG^n,X) \to 0.\end{aligned}$$

There is an equivalence

$$\operatorname{holim}_n \mathrm{Map}_G(EG^n,X) \cong \mathrm{Map}_G(\operatorname{hocolim}_n EG^n,X)$$

and $\operatorname{hocolim}_n EG_+^n$ is contractible. It follows that all the groups in the exact sequence are trivial, which is precisely the claim. $\square$

Lemma 2. *Assume X is a spectrum whose homotopy groups are finite groups. Then the spectral sequence* (1) *is strongly convergent.*

Proof. According to Theorem 7.3 in [6], it suffices to prove that the derived E^∞-term of (1) vanishes. This is clearly the case under the hypothesis on X. $\square$

2.2. The pairing

Let X be a ring spectrum. The mod n spectrum X/n of X is defined as the smash product $X \wedge S^0/n$ of X with the mod n Moore spectrum, and there is a homotopy equivalence $(X/n)^{hG} \simeq X^{hG} \wedge S^0/n$. By specialization we obtain the homotopy fixed point spectral sequence for X with mod n coefficients:

$$\mathrm{E}^2_{p,q}(X, S^0/n) = \mathrm{H}^{-p}(G; \pi_q/n(X)) \Rightarrow \pi_{p+q}/n(X^{hG}). \tag{3}$$

From [1] it is known that $X/2$ does not have a product structure in general, and even if there exists a product it might not be compatible with the G-action. Hence there is no reason to expect a Leibniz rule for the differentials in [3] for $n = 2$. To deal with this we will exploit the $S^0/4$-module structure

$$S^0/4 \wedge S^0/2 \to S^0/2$$

on $S^0/2$ constructed by Oka in [33] (called a $S^0/4$ premultiplication on $S^0/2$ in loc. cit.). There results a (left) unital pairing

$$m : \pi_*/4(X) \otimes \pi_*/2(X) \to \pi_*/2(X)$$

of mod 4 and mod 2 homotopy of X. On the spectral sequence level we obtain a natural module paring

$$\begin{aligned} \star :\ \mathrm{E}^r_{*,*}(X, S^0/4) \otimes \mathrm{E}^r_{*,*}(X, S^0/2) &\to \mathrm{E}^r_{*,*}(X, S^0/4 \wedge S^0/2) \\ &\to \mathrm{E}^r_{*,*}(X, S^0/2) \end{aligned} \tag{4}$$

of the E^r-pages of the mod 4 and the mod 2 version of (3). The word pairing will from now on refer to (4). For the pairing we do have a Leibniz rule. That is, for $x \in \mathrm{E}^r_{*,*}(X, S^0/4)$ and $y \in \mathrm{E}^r_{*,*}(X, S^0/2)$ we have

$$d^r(x \star y) = d^r(x) \star y + (-1)^{|x|} x \star d^r(\gamma).$$

Of course, the sign does not matter for spectra whose mod 2 homotopy groups have exponent two. For $r = 2$, the pairing

$$\star :\ \mathrm{H}^*(G; \pi_*/4(X)) \otimes \mathrm{H}^*(G; \pi_*/2(X)) \to \mathrm{H}^*(G; \pi_*/2(X))$$

is given as the composite of the cup-product in group cohomology and the coefficient pairing m.

Now we record a slick way of detecting permanent cycles in [3]. We use the convention that a class in a spectral sequence is an infinite cycle if all the differentials on that class are zero. A permanent cycle is an infinite cycle that is not in the image of any differential. The following result is well known.

Lemma 3. *Consider* (3) *for* $X = \mathrm{K}(R_E)$ *and* $G = C_2$. *Let x be a class in bidegree* $(0, q)$ *which is in the image of the canonically induced map from* $\mathrm{K}_q/n(R_F)$ *to* $\mathrm{K}_q/n(R_E)$. *Then x is a permanent cycle.*

3. Étale decent for the real numbers

In this section we verify that the real numbers satisfy étale descent with mod 2^v coefficients for all $v \geqslant 1$. This part is perhaps not very original, and contains no surprises. However, the calculation leading to the mentioned conclusion will be important in the proof of our main result. We also introduce notation for classes we will employ later.

3.1. The HFP spectral sequence for $\mathbf{R} \to \mathbf{C}$

Consider the homotopy fixed point spectral sequence:

$$\mathrm{E}^2_{p,q}(v) = \mathrm{H}^{-p}(C_2; \mathrm{K}_q/2^v(\mathbf{C})) \Rightarrow \pi_{p+q}/2^v(\mathrm{K}(\mathbf{C})^{hC_2}). \tag{5}$$

Recall from the Quillen–Lichtenbaum conjecture for algebraically closed fields, proven by Suslin [40], that the odd algebraic K-groups with finite coefficients of an algebraically closed field are trivial. Hence the E^3-page of (3.1.1) is identical to its E^2-page. From loc. cit. we have $\mathrm{K}_{2i}/2^v(\mathbf{C}) \cong \mathbf{Z}/2^v(i)$ for $i \geqslant 0$. First we compute the groups on the E^2-page.

Lemma 4. *For $p \geqslant 0$ we have*

$$\mathrm{H}^p(C_2; \mathrm{K}_{2i}/2^v(\mathbf{C})) = \begin{cases} \mathbf{Z}/2^v & \textit{for } p = 0 \textit{ and } i \textit{ even}, \\ \mathbf{Z}/2 & \textit{otherwise}. \end{cases}$$

Proof. The generator of $C_2 = \mathrm{Gal}(\mathbf{R} \to \mathbf{C})$ acts by multiplication by $(-1)^i$ on $\mathbf{Z}/2^v(i)$. So by applying $\mathrm{Hom}_{C_2}(-, \mathbf{Z}/2^v(i))$ to the standard periodic resolution of the trivial C_2-module $\mathbf{Z}$, we find the cocomplex

$$0 \to \mathbf{Z}/2^v \xrightarrow{1-(-1)^i} \mathbf{Z}/2^v \xrightarrow{1+(-1)^i} \mathbf{Z}/2^v \xrightarrow{1-(-1)^i} \cdots,$$

which computes the displayed cohomology groups. $\square$

To explicate permanent cycles in (5); consider the image of $\mathrm{K}_*/2^v(\mathbf{R})$ in $\mathrm{K}_*/2^v(\mathbf{C})$, cf. [41]. Let ko (resp. ku) denote the (-1)-connected cover of the real (resp. complex) topological K-theory spectrum KO (resp. KU).

Theorem 6 (Suslin). *The natural maps* $\mathrm{K}(\mathbf{R}) \to \mathrm{ko}$ *and* $\mathrm{K}(\mathbf{C}) \to \mathrm{ku}$ *induce homotopy equivalences after two-adic completion.*

Corollary 5. *There is the following table for $v \geqslant 2$:*

$*$ mod 8	1	2	3	4	8
$\mathrm{K}_*/2^v(\mathbf{R})$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2^v$	$\mathbf{Z}/2^v$
$\mathrm{K}_*/2(\mathbf{R})$	$\mathbf{Z}/2$	$\mathbf{Z}/4$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$

The group $\mathrm{K}_*/2^v(\mathbf{R})$ *is zero for* $* \equiv 5,6,7 \bmod 8$ *and all* $v \geqslant 1$. *The natural map*

$$\mathrm{K}_*/2^v(\mathbf{R}) \to \mathrm{K}_*/2^v(\mathbf{C})^{C_2}$$

(*the target group is given by Lemma* 3.1.2) *is the identity if* $* \equiv 8 \bmod 8$, *surjective if* $* \equiv 2 \bmod 8$, *multiplication by two if* $* \equiv 4 \bmod 8$, *and zero otherwise.*

Proof. We remark that the last claim may be checked by using the cofibration sequence of spectra $\Sigma\mathrm{ko} \to \mathrm{ko} \xrightarrow{c} \mathrm{ku}$. Here c denotes the complexification map, and the first map is multiplication by the complex Hopf map. □

Let S^0 be the sphere spectrum. We write η_v for the mod 2^v reduction of the generator η of $\pi_1(S^0)$, and v_1 for the mod 2^v reduction of the generator of $\pi_2(\mathrm{K}U)$ that induces the Bott periodicity in complex topological K-theory.

Lemma 6. *In the spectral sequence* (5), *there are the following permanent cycles.*

(i) *The unit element* 1 *in bidegree* $(0,0)$, *the class* $2^{v-1}v_1$ *in bidegree* $(0,2)$ *and the class* v_1^4 *in bidegree* $(0,8)$.
(ii) *The class represented by* η_v *in bidegree* $(-1,2)$.

Proof. Part (1) is immediate from Theorem 6 and Lemma 2.2.3. For part (2) we note that $\mathrm{Map}_{C_2}(EC^{(1)}_{2+}, \mathrm{KU})$ is the spectrum KSC of self-conjugate topological K-theory. There are unique KO-module maps KO → KSC → KU respecting the unit maps, and whose composite KO → KU equals the complexification map. It is classical, and may be checked from the maps above, that $\pi_1/2^v(\mathrm{KO}) \to \pi_1/2^v(\mathrm{KSC})$ is an isomorphism. □

Recall the classes η and $\tilde{\sigma}_{16} \in \pi_8/16(S^0)$ in stable homotopy groups of spheres. Under the unit map $S^0 \to \mathrm{K}(R_{\mathbf{Q}})$, these classes have images $\eta \in \mathrm{K}_1(R_{\mathbf{Q}})$ and $v_1^4 \in \mathrm{K}_8/16(R_{\mathbf{Q}})$. Under the canonical map $\mathrm{K}(R_{\mathbf{Q}}) \to \mathrm{K}(R_{\mathbf{Q}(\sqrt{-1})})^{hC_2}$, these classes are represented in mod 4 homotopy as the generator η_2 of $\mathrm{H}^1(C_2; \mathrm{K}_2/4(R_{\mathbf{Q}(\sqrt{-1})}))$ and as the generator v_1^4 of $\mathrm{H}^0(C_2; \mathrm{K}_8/4(R_{\mathbf{Q}(\sqrt{-1})}))$. The classes η_2 and v_1^4 are therefore permanent cycles in the homotopy fixed point spectral sequence for the extension $R_{\mathbf{Q}} \to R_{\mathbf{Q}(\sqrt{-1})}$. Likewise for the mod 2 classes. These classes map by naturality to infinite cycles in the homotopy fixed point spectral sequence for the extension $R_F \to R_E$ for any real number field F. We keep, independently of the number field, the notation η_2 and v_1^4 for those infinite cycles. In fact, both η_2 and v_1^4 are permanent cycles (both classes are non-zero because the composition $R_{\mathbf{Q}(\sqrt{-1})} \to \mathbf{C}$ induces an isomorphism on the E^2-page in even rows, and neither of them can be hit by differentials for bidegree reasons). By starting with the generator of $\mathrm{H}^2(C_2; \mathrm{K}_0/4(R_{\mathbf{Q}(\sqrt{-1})}))$ we construct the class y_2 for any F and also $\mathbf{C}$ by the same device as above.

To proceed with the other differentials we will exploit the cup-product map in group cohomology. Let T denote the generator of C_2. The following observations will be applied several times, since it allows to conclude that certain differentials replicate in the homotopy fixed point spectral sequence. Recall the computation of $\mathrm{K}_*/2(R_E)$ given in the introduction.

Lemma 7. *Let M_q be $\mathrm{K}_q/2(R_E)$ (assume $q \geqslant 1$ in this case) or $\mathrm{K}_q/2(\mathbf{C})$. Let ρ_i^p be the naturally induced map $\mathrm{H}^p(C_2, \mathbf{Z}/4(i)) \to \mathrm{H}^p(C_2, \mathbf{Z}/2(i))$. Then*

(i) *The maps ρ_1^1 and ρ_0^2 are isomorphisms, while ρ_4^0 is a surjection.*
(ii) *$\eta_2 \cup - : \mathrm{H}^p(C_2, M_q) \to \mathrm{H}^{p+1}(C_2, M_{q+2})$ is surjective for $p = 0$, and an isomorphism for $p \geqslant 1$.*
(iii) *Cup-product with the mod 4 class $v_1^4 \cup - : \mathrm{H}^p(C_2, M_q) \to \mathrm{H}^p(C_2, M_{q+8})$ is an isomorphism.*
(iv) *$y_2 \cup - : \mathrm{H}^p(C_2, M_q) \to \mathrm{H}^{p+2}(C_2, M_q)$ is surjective for $p = 0$, and bijective for $p \geqslant 1$.*

Proof. (i) These claims can be checked by comparing the complexes for $v = 1$ and 2 in the proof of Lemma 3.1.2.

The rest follows from (i) and a cup-product calculation using the mod 2 classes. Alternatively we could cup with the mod 4 classes directly. Here, η_2 represents a cocycle $\mathbf{Z}[C_2] \xrightarrow{1 \mapsto 1, T \mapsto -1} \mathbf{Z}/4(1)$, and the mod 4 class v_1^4 in bidegree (0,8) of (5) represents a cocycle $\mathbf{Z}[C_2] \xrightarrow{1, T \mapsto 1} \mathbf{Z}/4(4)$. □

We point out that ρ_1^0 is zero. This explains why we can not employ the permanent cycle in bidegree (0,2) to replicate the differentials vertically.

Lemma 8. *For the spectral sequence* (5) *we have*:

(i) *The d^3-differential $\mathrm{E}^3_{0,4}(1) \to \mathrm{E}^3_{-3,6}(1)$ is an isomorphism.*
(ii) *The d^3-differential $\mathrm{E}^3_{0,6}(1) \to \mathrm{E}^3_{-3,8}(1)$ is an isomorphism.*
(iii) *The class $\eta_1^2 := \eta_2 \star \eta_1$ in bidegree $(-2,4)$ is a permanent cycle.*
(iv) *The class $\eta_1 v_1 := \eta_2 \star v_1$ in bidegree $(-1,4)$ and the class $\eta_1^2 v_1 := \eta_2^2 \star v_1$ in bidegree $(-2,6)$ are permanent cycles.*

Proof. From Lemma 7; the class $\eta_1^3 := \eta_2^2 \star \eta_1$ represents the non-trivial element of $\mathrm{E}^3_{-3,6}(1)$. Hence in the mod 2 spectral sequence we have

$$d^r(\eta_1^3) = d^r(\eta_2^2) \star \eta_1 + \eta_2^2 \star d^r(\eta_1).$$

Here η_1 is a permanent cycle from Lemma 6, and η_2^2 is an infinite cycle in $\mathrm{E}^3_{-2,4}(2)$. This proves that η_1^3 is an infinite cycle. The same argument shows that η_1^2 is a permanent cycle for bidegree reasons. However, the class η_1^3 is not a permanent cycle since the unit map $S^0 \to \mathrm{K}(\mathbf{C})^{hC_2}$ factors through $\mathrm{K}(\mathbf{R})$ where $\eta^3 \in \pi_3(S^0)$ has a trivial image. The only possible non-trivial differential with target $\mathrm{E}^r_{-3,6}(1)$ where $r \geqslant 2$ has source $\mathrm{E}^3_{0,4}(1)$, and that differential must therefore be an isomorphism. Done with Parts (i) and (iii).

Multiplication by the class η_2 is a surjection on the E^3-page of (5), so the non-trivial class in bidegree $(-3, 8)$ is in the image of the map $\pi_5/2(S^0) \to \pi_5/2(\mathrm{K}(\mathbf{C})^{hC_2})$ whose source is known to be the trivial group. The same class is an infinite cycle by the pairing argument above, and whence (ii).

Part (iv) is clear from Lemma 6 and the pairing argument. □

Theorem 7 (Étale descent for the reals). *The canonical map $\mathrm{K}(\mathbf{R}) \to \mathrm{K}(\mathbf{C})^{hC_2}$ induces a mod 2^v isomorphism in non-negative dimensions for all $v \geqslant 1$.*

Proof. The permanent cycle η_2 replicates the differentials up and to the left, while the permanent cycle v_1^4 replicates the differentials vertically. This is straight forward from the previous lemmas. We get the following picture of the non-trivial columns of the 8-periodic $E^4 = E^\infty$-page with the origin in the bottom right corner:

0	0	$\mathbf{Z}/2(v_1^4)$
0	0	0
$\mathbf{Z}/2(\eta_1^2 v_1)$	0	0
0	0	0
$\mathbf{Z}/2(\eta_1^2)$	$\mathbf{Z}/2(\eta_1 v_1)$	0
0	0	0
0	$\mathbf{Z}/2(\eta_1)$	$\mathbf{Z}/2(v_1)$
0	0	0
0	0	$\mathbf{Z}/2(1)$

Note that all the classes on the axis $E_{0,q}^\infty(1)$ are in the image of $K/2(\mathbf{R})$ from Corollary 5. The rest of the classes can be written as η_2 or η_2^2 times a class from the axis $E_{0,q}^\infty(1)$. From these remarks and the table in Corollary 5, we deduce that the natural map

$$K/2(\mathbf{R}) \to K/2(\mathbf{C})^{hC_2}$$

is an equivalence. □

4. Étale descent for real number fields

In this section we give a proof for the strong Quillen–Lichtenbaum conjecture for real number fields at the prime two, by computing the homotopy fixed point spectral sequence of the spectrum $K/2(R_E)$ for the group of order two. In the first part of the proof we analyse the Lyndon–Hochschild–Serre spectral sequences for the extension $R_F \to R_E$ for mod 2 étale cohomology and mod 2 positive étale cohomology. Those spectral sequences govern all the arithmetic information we will need later on. In Section 4.1 we translate this into a statement about algebraic K-theory by utilizing naturality of the extended Bloch–Lichtenbaum spectral sequence constructed by Levine [23]. In effect, we obtain a description of the image of $K_*/2(R_F)$ in $K_*/2(R_E)$, and hence we know some permanent cycles in the mod 2 homotopy fixed point spectral sequence. The rest of the calculation follows by the same scheme as for the reals.

4.1. The Lyndon–Hochschild–Serre spectral sequence

We start with some general remarks. Let $Y \to X$ be a finite Galois covering of schemes with Galois group G. From [49], the Galois group acts on the mod 2 étale cohomology groups $H^*_{\text{ét}}(Y;\mathbf{Z}/2)$ of Y, and there exists a first quadrant cohomological spectral sequence

$$E_2^{p,q}(Y;\mathbf{Z}/2) = H^p(G, H^q_{\text{ét}}(Y;\mathbf{Z}/2)) \Rightarrow H^{p+q}_{\text{ét}}(X;\mathbf{Z}/2). \quad (6)$$

We call (6) the LHS spectral sequence from now on. By specialization, there exists a LHS spectral sequence for the extension $R_F \to R_E$ since $F \to E$ is unramified away from the infinite primes and the dyadic primes (primes lying above the rational prime (2)). The associated exact sequence of terms of low degree is

$$0 \to \mathrm{H}^1(C_2, \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2)) \to \mathrm{H}^1_{\text{ét}}(R_F; \mathbf{Z}/2) \to \mathrm{H}^0(C_2, \mathrm{H}^1_{\text{ét}}(R_E; \mathbf{Z}/2))$$
$$\xrightarrow{d_2^{0,1}} \mathrm{H}^2(C_2, \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2)) \to \mathrm{H}^2_{\text{ét}}(R_F; \mathbf{Z}/2). \tag{7}$$

A real embedding of F, recall that $r_1 \geqslant 1$, induces by functoriality of LHS a homomorphism of spectral sequences

$$\mathrm{H}^p(C_2, \mathrm{H}^q_{\text{ét}}(R_E; \mathbf{Z}/2)) \xrightarrow{i_{p,q}} \mathrm{H}^p(C_2, \mathrm{H}^q_{\text{ét}}(\mathbf{C}; \mathbf{Z}/2)).$$

There are no non-trivial differentials in the LHS spectral sequence for the extension $\mathbf{R} \to \mathbf{C}$, so the E_2-page of (6) equals the E_∞-page of (6) in this case. Note that the map $i_{p,0}$ is an isomorphism. Hence all the differentials $d_r^{p-r,r-1}$ in the LHS spectral sequence for $R_F \to R_E$ are trivial, cp. the diagram:

$$\begin{tikzcd} \mathrm{E}_r^{p-r,r-1}(R_E; \mathbf{Z}/2) \arrow[r] \arrow[d, "d_r^{p-r,r-1}"'] & \mathrm{E}_r^{p-r,r-1}(\mathbf{C}; \mathbf{Z}/2) \arrow[d, "d_r^{p-r,r-1}"] \\ \mathrm{E}_r^{p,0}(R_E; \mathbf{Z}/2) \arrow[r, "\cong"] & \mathrm{E}_r^{p,0}(\mathbf{C}; \mathbf{Z}/2) \end{tikzcd}$$

Since E is totally imaginary, $\mathrm{H}^q_{\text{ét}}(R_E; \mathbf{Z}/2) = 0$ for $q \geqslant 3$ by a theorem of Tate [44]. This implies

Lemma 9. *The only possibly non-trivial differentials in* (6) *for the extension* $R_F \to R_E$ *are the* d_2*-differentials with source* $\mathrm{E}_2^{p,2}$ *for some* $p \geqslant 0$. *Hence* $\mathrm{E}_2^{p,0} \cong \mathrm{E}_\infty^{p,0}$ *for all* $p \geqslant 0$, $\mathrm{E}_2^{0,1} \cong \mathrm{E}_\infty^{0,1}$ *and* $\mathrm{E}_2^{1,1} \cong \mathrm{E}_\infty^{1,1}$. *Moreover, the exact sequence* (7) *simplifies to a short exact sequence*

$$0 \to \mathrm{H}^1(C_2, \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2)) \to \mathrm{H}^1_{\text{ét}}(R_F; \mathbf{Z}/2) \to \mathrm{H}^0(C_2, \mathrm{H}^1_{\text{ét}}(R_E; \mathbf{Z}/2)) \to 0.$$

Remark 10. Essentially the same result can be found for totally real number fields in Lemme 1.5 [32].

In the following we write s_F for the number of dyadic primes in F, t_F for the two rank of the Picard group $\mathrm{Pic}(R_F)$ of R_F, and t_F^+ for the two rank of the narrow Picard group $\mathrm{Pic}_+(R_F)$ of R_F. Likewise we write s_E for the number of dyadic primes in E, and t_E for the two rank of the Picard group $\mathrm{Pic}(R_E)$ of R_E. Let T be the generator of $C_2 = \mathrm{Gal}(R_F \to R_E)$.

Lemma 11. *Let* $p \geqslant 0$. *Then*

$$\mathrm{rk}_2\, \mathrm{H}^p(C_2, \mathrm{H}^1_{\text{ét}}(R_E; \mathbf{Z}/2)) = \begin{cases} r_1 + r_2 + s_F + t_F - 1 & \textit{for } p = 0, \\ r_1 + 2(s_F + t_F) - (s_E + t_E) - 2 & \textit{for } p \geqslant 1. \end{cases}$$

Proof. The isomorphism $\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2) \cong R_F^*/2 \oplus_2 \mathrm{Pic}(R_F)$ and Lemma 9 give the result for $p=0$. In the cocomplex

$$\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2) \stackrel{1-T}{\to} \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2) \stackrel{1+T}{\to} \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2) \stackrel{1-T}{\to} \cdots$$

computing the C_2 cohomology groups of $\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)$ we have $\ker(1-T) \cong \ker(1+T) \cong \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)^{C_2}$ since $\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)$ has exponent two. Note that the extension $F \to E$ is of degree two, so E has precisely r_1+2r_2 pairs of complex embeddings. Hence the two rank of $\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)$ equals $r_1+2r_2+s_E+t_E$, and therefore $\mathrm{rk}_2\,\mathrm{im}(1\pm T) = r_2+(s_E+t_E)-(s_F+t_F)+1$. □

Lemma 12. *Let $d_2^{p,2}: \mathrm{H}^p(C_2,\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)) \to \mathrm{H}^{p+2}(C_2,\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2))$ be the d_2-differential in the LHS spectral sequence for the extension $R_F \to R_E$. Then*

$$\mathrm{rk}_2\ker(d_2^{p,2}) = \begin{cases} (s_E+t_E)-(s_F+t_F) & \text{for } p=0,\\ t_F^+ - t_F & \text{for } p\geqslant 1,\end{cases}$$

and $\mathrm{rk}_2\,\mathrm{im}(d_2^{p,2}) = 2s_F+t_F^++t_F-(s_E+t_E)-1$ *for all p. Hence* $\mathrm{rk}_2\,\mathrm{E}_\infty^{p,1} = r_1+t_F-t_F^+-1$ *for all $p\geqslant 2$, and the image of $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)$ in $\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$ has two rank $(s_E+t_E)-(s_F+t_F)$.*

For $p\geqslant 0$ we have

$$\mathrm{rk}_2\,\mathrm{H}^p(C_2,\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)) = \begin{cases} s_F+t_F^+-1 & \text{for } p=0,\\ 2(s_F+t_F^+)-(s_E+t_E)-1 & \text{for } p\geqslant 1.\end{cases}$$

Proof. The LHS spectral sequence implies that

$$\mathrm{rk}_2\ker(d_2^{0,2}) + \mathrm{rk}_2\,\mathrm{E}_2^{1,1} + \mathrm{rk}_2\,\mathrm{E}_2^{2,0} = \mathrm{rk}_2\,\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2).$$

From above, $\mathrm{rk}_2\,\mathrm{E}_2^{1,1} = r_1+2(s_F+t_F)-(s_E+t_E)-2$, and that $\mathrm{rk}_2\,\mathrm{E}_2^{2,0}=1$. The two rank of $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)$ is $r_1+s_F+t_F-1$, and this determines $\mathrm{rk}_2\ker(d_2^{0,2})$.

We postpone the proof of the equality $\mathrm{rk}_2\,\mathrm{H}^0(C_2,\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)) = s_F+t_F^+-1$ until Lemma 14. Taking this for granted we have $\mathrm{rk}_2\ker(1\pm T) = s_F+t_F^+-1$, for $1\pm T: \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2) \to \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$, and we may proceed as in Lemma 11. The two rank of $\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$ equals s_E+t_E-1, and it follows that $\mathrm{rk}_2\,\mathrm{im}(1\pm T) = (s_E+t_E)-(s_F+t_F^+)$.

The other claims are now easy consequences, of what we have already proved or assumed above, since $\mathrm{H}^q_{\text{ét}}(R_F;\mathbf{Z}/2)$ is an elementary Abelian two-group of rank r_1 for all $q\geqslant 3$ according to [44]. □

I have not managed to give a proof of the claimed two-rank equality without using positive étale cohomology. This is our next theme.

A real embedding of F gives a map $F^* \to \mathbf{R}^*/2$ which detects the sign of the units in F. The real embeddings of F assemble to define the sign map of F

$$\sigma: F^* \to \bigoplus^{r_1} \mathbf{Z}/2.$$

The approximation theorem says that σ is surjective. If we restrict the source of σ to the units of R_F, then it is no longer true in general that σ is surjective. Consider the mod 2 reduction

$$\sigma/2: R_F^*/2 \to \bigoplus^{r_1} \mathbf{Z}/2$$

of σ restricted to the units of R_F. The image of the latter map can be identified with the image of the naturally induced map

$$\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2)\to\overset{r_1}{\oplus}\mathrm{H}^1_{\text{ét}}(\mathbf{R};\mathbf{Z}/2)$$

which we, by abuse of notation, also denote by $\sigma/2$. Lemma 7.6 in [36] shows that

$$\mathrm{rk}_2\,\mathrm{im}\,\sigma/2 = r_1 + t_F - t_F^+. \tag{8}$$

Next, we use the above to compare mod 2 positive étale cohomology groups, indicated by the lower index $+$, with ordinary mod 2 étale cohomology groups. The positive étale cohomology groups of R_F are defined in [9]. From loc. cit. we have, with a different indexing, the exact sequence

$$\begin{aligned}&0\to \mathrm{H}^0_{\text{ét}}(R_F;\mathbf{Z}/2)\to \oplus^{r_1+r_2}\mathbf{Z}/2\to \mathrm{H}^1_+(R_F;\mathbf{Z}/2)\to \mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2)\\ &\quad\xrightarrow{\sigma/2}\oplus^{r_1}\mathbf{Z}/2\to \mathrm{H}^2_+(R_F;\mathbf{Z}/2)\to \mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)\to \oplus^{r_1}\mathbf{Z}/2\to 0.\end{aligned} \tag{9}$$

Lemma 13. *For $q \geqslant 0$ there are the following two-rank formulas*:

$$\mathrm{rk}_2\,\mathrm{H}^q_+(R_F;\mathbf{Z}/2)=\begin{cases} r_1+2r_2+s_F+t_F^+-1 & \text{for } q=1,\\ s_F+t_F^+-1 & \text{for } q=2,\\ 0 & \text{otherwise.}\end{cases}$$

$$\mathrm{rk}_2\,\mathrm{im}(\mathrm{H}^q_+(R_F;\mathbf{Z}/2)\to \mathrm{H}^q_{\text{ét}}(R_F;\mathbf{Z}/2))=\begin{cases} r_2+s_F+t_F^+ & \text{for } q=1,\\ s_F+t_F-1 & \text{for } q=2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Follows from (8) and (9) by counting. □

The Lyndon–Hochschild–Serre spectral sequence for the étale sheaf that defines the mod 2 positive étale cohomology groups is a first quadrant cohomological spectral sequence

$$\mathrm{E}_2^{p,q}(R_E;\mathbf{Z}/2)_+ = \mathrm{H}^p(C_2,\mathrm{H}^q_+(R_E;\mathbf{Z}/2)) \Rightarrow \mathrm{H}^{p+q}_+(R_F;\mathbf{Z}/2). \tag{10}$$

We call (10) the positive LHS spectral sequence from now on.

Lemma 14. *Let $d_2^{0,2}$: $\mathrm{H}^0(C_2,\mathrm{H}^2_+(R_E;\mathbf{Z}/2))\to \mathrm{H}^2(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))$ be the d_2-differential in the positive LHS spectral sequence for the extension $R_F\to R_E$. Then*

$$\mathrm{rk}_2\ker(d_2^{0,2}) = (s_E+t_E)-(s_F+t_F^+).$$

Hence the image of the canonical map $\mathrm{H}^2_+(R_F;\mathbf{Z}/2)\to \mathrm{H}^2_+(R_E;\mathbf{Z}/2)$ has two rank $(s_E+t_E)-(s_F+t_F^+)$, while its cokernel has two rank $s_F+t_F^+-1$. For $p\geqslant 0$ it follows that

$$\mathrm{rk}_2\,\mathrm{H}^p(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))=\begin{cases} r_1+2r_2+s_F+t_F^+-1 & \text{for } p=0,\\ 2(s_F+t_F^+)-(s_E+t_E)-1 & \text{for } p\geqslant 1\end{cases}$$

The map $\mathrm{H}^2_+(R_E;\mathbf{Z}/2) \to \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$ *is an isomorphism, and there is the equality* $\mathrm{rk}_2\,\mathrm{H}^0(C_2,\mathrm{H}^2_+(R_E;\mathbf{Z}/2)) = s_F + t_F^+ - 1$.

Proof. The only possibly non-trivial groups in (10) are located in bidegrees (p,q) with $q=1$ or 2, cf. Lemma 13. Hence $\mathrm{H}^0(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))$ is isomorphic to $\mathrm{H}^1_+(R_F;\mathbf{Z}/2)$. Lemma 13 and the argument for Lemma 11 imply the two-rank formula for $\mathrm{H}^p(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))$.

From the E^3-page of (10) we read off the equalities

$$\begin{aligned}\mathrm{rk}_2\ker(d_2^{0,2}) &= \mathrm{rk}_2\,\mathrm{H}^2_+(R_F;\mathbf{Z}/2) - \mathrm{rk}_2\,\mathrm{H}^1(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))\\ &= (s_F + t_F^+ - 1) - (2(s_F+t_F^+) - (s_E+t_E) - 1)\\ &= (s_E+t_E) - (s_F+t_F^+).\end{aligned}$$

The differential $d_2^{0,2}$ is surjective since $\mathrm{H}^3_+(R_F;\mathbf{Z}/2)$ is the trivial group, and we find

$$\begin{aligned}\mathrm{rk}_2\,\mathrm{H}^0(C_2,\mathrm{H}^2_+(R_E;\mathbf{Z}/2)) &= \mathrm{rk}_2\ker(d_2^{0,2}) + \mathrm{rk}_2\,\mathrm{H}^2(C_2,\mathrm{H}^1_+(R_E;\mathbf{Z}/2))\\ &= (s_E+t_E) - (s_F+t_F^+) + 2(s_F+t_F^+) - (s_E+t_F) - 1\\ &= s_F + t_F^+ - 1.\end{aligned}$$

Note that the map from $\mathrm{H}^2_+(R_E;\mathbf{Z}/2)$ to $\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$ is an isomorphism by (9). This justifies the calculation announced in Lemma 12. □

4.2. On the map $\mathrm{K}_*/2(R_F) \to \mathrm{K}_*/2(R_E)$

In this subsection we will identify the image of $\mathrm{K}_*/2(R_F)$ in $\mathrm{K}_*/2(R_E)$. This will be accomplished by comparing with étale and positive étale cohomology. For this to work out, we need to know that the abstract comparison between the étale and positive étale cohomology of R_F and the mod 2 homotopy groups of $\mathrm{K}(R_F)$ behaves naturally. This is plain from the work of Levine in [23]. His extended Bloch–Lichtenbaum spectral sequence for number rings has the required naturality properties.

Proposition 15. *For the natural map* $\phi_n\colon \mathrm{K}_n/2(R_F) \to \mathrm{K}_n/2(R_E)^{C_2}$ *and* $n \geqslant 1$ *we have*

$$\mathrm{rk}_2\,\mathrm{im}(\phi_n) = \begin{cases} r_1 + r_2 + s_F + t_F - 1 & \text{for } n \equiv 1,3 \bmod 8,\\ (s_E+t_E) - (s_F+t_F) + 1 & \text{for } n \equiv 2 \bmod 8,\\ (s_E+t_E) - (s_F+t_F) & \text{for } n \equiv 4 \bmod 8,\\ r_2 + s_F + t_F^+ & \text{for } n \equiv 5,7 \bmod 8,\\ (s_E+t_E) - (s_F+t_F^+) & \text{for } n \equiv 6 \bmod 8,\\ (s_E+t_E) - (s_F+t_F^+) + 1 & \text{for } n \equiv 8 \bmod 8.\end{cases}$$

Proof. All the identifications we will make of mod 2 algebraic K-groups can be found in Theorem 7.8 in [36]. The diagrams we will display commute due to naturality of the extended Bloch–Lichtenbaum spectral sequence [23]. It can be written as a homological, first quadrant strongly convergent algebra spectral sequence

$$\mathrm{E}^2_{p,q}(R_F;\mathbf{Z}/2)=\mathrm{H}^{q-p}_{\text{ét}}(R_F;\mathbf{Z}/2)\Rightarrow \mathrm{K}_{p+q}/2(R_E) \tag{11}$$

converging to the mod 2 algebraic K-groups of R_F. It was left partially undecided in [36], but all the non-trivial differentials in (11) are d^2-differentials. The filtrations we will consider below are the filtrations giving rise to (11).

For $n\equiv 1,3 \bmod 8$ the edge maps in (11) induce the diagram

$$\begin{tikzcd}
\mathrm{K}_n/2(R_F) \arrow[r] \arrow[d] & \mathrm{K}_n/2(R_E)^{C_2} \arrow[d, "\cong"] \\
\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2) \arrow[r] & \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)^{C_2}
\end{tikzcd}$$

From Lemma 9; the lower map is a split surjection of $\mathbf{Z}/2$-modules and its kernel has order two. Lemma 11 says that

$$\mathrm{rk}_2\,\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)^{C_2}=r_1+r_2+s_F+t_F-1.$$

The left vertical map is surjective. Its kernel is the trivial group for $n\equiv 1 \bmod 8$, and an elementary Abelian two-group of rank r_1-1 for $n\equiv 3 \bmod 8$.

For $n\equiv 5 \bmod 8$ there is the exact sequence

$$\begin{array}{llll}
0 & \to & (\mathbf{Z}/2)^{r_1-1} & \to & \mathrm{K}_n/2(R_F) \\
 & \to & \mathrm{im}(\mathrm{H}^1_+(R_F;\mathbf{Z}/2) & \to & \mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2))\to 0.
\end{array}$$

Consider now the diagram

$$\begin{tikzcd}
\mathrm{H}^1(C_2,\mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2)) \arrow[r] & \mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2) \arrow[r] \arrow[d, "\sigma/2"] & \mathrm{H}^0(C_2,\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)) \\
 & \bigoplus^{r_1}\mathrm{H}^1_{\text{ét}}(\mathbf{R};\mathbf{Z}/2) &
\end{tikzcd}$$

where the horizontal sequence comes from Lemma 9 and the vertical sequence comes from (9). The composite map $\mathrm{H}^1(C_2,\mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2))\to\oplus^{r_1}\mathrm{H}^1_{\text{ét}}(\mathbf{R};\mathbf{Z}/2)$ is the diagonal map, so it follows that $\mathrm{im}(\mathrm{H}^1_+(R_F;\mathbf{Z}/2)\to\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2))$ injects into $\mathrm{H}^0(C_2,\mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2))$. The claim follows from the diagram

$$\begin{tikzcd}
\mathrm{K}_n/2(R_F) \arrow[r] \arrow[d] & \mathrm{K}_n/2(R_E)^{C_2} \arrow[d, "\cong"] \\
\mathrm{im}(\mathrm{H}^1_+(R_F;\mathbf{Z}/2)\to\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2)) \arrow[r] & \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)^{C_2}
\end{tikzcd}$$

and Lemma 13. For $n\equiv 7 \bmod 8$ we can identify $\mathrm{K}_n/2(R_F)$ with the image of $\mathrm{H}^1_+(R_F;\mathbf{Z}/2)$ in $\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2)$, and the claim follows by the same argument as for $n\equiv 5 \bmod 8$.

Let n be even from now on. For $n \equiv 2 \bmod 8$ there is the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2) \arrow[r] \arrow[d] & \mathrm{K}_n/2(R_F) \arrow[r] \arrow[d] & \mathrm{H}^0_{\text{ét}}(R_F;\mathbf{Z}/2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2) \arrow[r] & \mathrm{K}_n/2(R_E) \arrow[r] & \mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

The right vertical map is an isomorphism, and the image of the left vertical map was calculated in Lemma 12. For $n \equiv 4 \bmod 8$ we read off from (11) the exact sequence

$$0 \to (\mathbf{Z}/2)^{r_1-1} \to \mathrm{K}_n/2(R_F) \to \mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2) \to 0.$$

By a filtration argument, cp. the next case, it follows that the image of ϕ_n equals the image of $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)$ in $\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$; which is known from Lemma 12. For $n \equiv 6 \bmod 8$ we have the exact sequence

$$\begin{array}{lllll}
\mathrm{H}^1_{\text{ét}}(R_F;\mathbf{Z}/2) & \overset{\sigma/2}{\to} & \oplus^{r_1}\mathbf{Z}/2 & \to & \mathrm{K}_n/2(R_F) \quad \to \\
& \to & \mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2) & \to & \oplus^{r_1}\mathbf{Z}/2 \quad \to 0.
\end{array}$$

To find the image of ϕ_n we consider the diagram of increasing filtrations:

$$\begin{tikzcd}
\mathrm{coker}(\sigma/2) \subset & \mathrm{F}_{\frac{n}{2}-1}\mathrm{K}_n/2(R_F) = \arrow[d] & \mathrm{F}_{\frac{n}{2}}\mathrm{K}_n/2(R_F) = \arrow[d] & \mathrm{K}_n/2(R_F) \arrow[d] \\
0 \subset & \mathrm{F}_{\frac{n}{2}-1}\mathrm{K}_n/2(R_E) \subset & \mathrm{F}_{\frac{n}{2}}\mathrm{K}_n/2(R_E) = & \mathrm{K}_n/2(R_E)
\end{tikzcd}$$

Here $\mathrm{F}_{n/2-1}\mathrm{K}_n/2(R_F)$ surjects onto $\mathrm{E}^\infty_{n/2-1,n/2+1}(R_F;\mathbf{Z}/2)$ since $\mathrm{E}^\infty_{n/2,n/2}(R_F;\mathbf{Z}/2)$ is the trivial group by the differential structure. Hence $\mathrm{E}^\infty_{n/2-1,n/2+1}(R_F;\mathbf{Z}/2)$ is the kernel of the map $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2) \to \oplus^{r_1}\mathbf{Z}/2$, which by (9) is isomorphic to the image of $\mathrm{H}^2_+(R_F;\mathbf{Z}/2)$ in $\mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)$. That group is an elementary Abelian two group of rank $(s_E + t_F) - (s_F + t_F^+)$ according to Lemma 4.1.11.

Now for the last case $n \equiv 8 \bmod 8$. By replacing $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)$ with the image of $\mathrm{H}^2_+(R_F;\mathbf{Z}/2)$ in $\mathrm{H}^2_{\text{ét}}(R_F;\mathbf{Z}/2)$ we have the same diagram depicted above for $n \equiv 2 \bmod 8$, and the claim follows by Lemma 4.1.11. □

4.3. The HFP spectral sequence for $R_F \to R_E$

In this subsection we make the announced calculation of the spectral sequence

$$\mathrm{E}^2_{p,q}(R_E;\mathbf{Z}/2) = \mathrm{H}^{-p}(C_2, \mathrm{K}_q/2(R_E)) \Rightarrow \pi_{p+q}(\mathrm{K}/2(R_E)^{hC_2}) \tag{12}$$

for $q \geqslant 1$. In the introduction we noted that

$$\mathrm{E}^2_{p,q}(R_E;\mathbf{Z}/2) = \begin{cases} \mathrm{H}^{-p}(C_2, \mathrm{H}^1_{\text{ét}}(R_E;\mathbf{Z}/2)) & \text{for } q \text{ odd,} \\ \mathrm{H}^{-p}(C_2, \mathrm{H}^2_{\text{ét}}(R_E;\mathbf{Z}/2)) & \\ \oplus & \\ \mathrm{H}^{-p}(C_2, \mathrm{H}^0_{\text{ét}}(R_E;\mathbf{Z}/2)) & \text{for } q \text{ even} \end{cases}$$

The groups on the E^2-page of (12) are given by Lemmas 11 and 12. The splitting allows us to make a kind of separation of variables when it comes to the differentials in (12). We can say something more about (12) without computing a single differential. Recall the notation from Lemma 7.

Lemma 16. *For $q \geqslant 1$ we have*

(i) *Multiplication by η_2 induces a map $\mathrm{E}^r_{p,q}(R_E;\mathbf{Z}/2) \to \mathrm{E}^r_{p-1,q+2}(R_E;\mathbf{Z}/2)$ which is*

- $r=2$; *a surjection for* $p=0$, *a bijection for* $p \leqslant -1$.
- $r=3$; *a surjection for* $p \leqslant 0$.

(ii) *Multiplication by v_1^4 induces a map $\mathrm{E}^r_{p,q}(R_E;\mathbf{Z}/2) \to \mathrm{E}^r_{p,q+8}(R_E;\mathbf{Z}/2)$ which is a bijection for all $p+q \geqslant 0$. We refer to this result as eight periodicity from now on.*

Proof. We prove (i). The claim for $r=2$ is Lemma 7. It follows that η_2 induces an isomorphism from $\operatorname{im} d^2_{p,q}$ to $\operatorname{im} d^2_{p-1,q+2}$ for all $p \leqslant 0$, cp. the diagram

$$\begin{tikzcd}
\mathrm{E}^2_{p,q}(R_E;\mathbf{Z}/2) \arrow[r, "d^2_{p,q}"] \arrow[d, "\eta_2"'] & \mathrm{E}^2_{p-2,q+1}(R_E;\mathbf{Z}/2) \arrow[d, "\eta_2"] \\
\mathrm{E}^2_{p-1,q+2}(R_E;\mathbf{Z}/2) \arrow[r, "d^2_{p-1,q+2}"] & \mathrm{E}^2_{p-3,q+3}(R_E;\mathbf{Z}/2)
\end{tikzcd}$$

where the left vertical map is surjective, and the right vertical map is bijective. The snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker d^2_{p,q} \arrow[r] \arrow[d, "\eta_2"] & \mathrm{E}^2_{p,q}(R_E;\mathbf{Z}/2) \arrow[r] \arrow[d, "\eta_2"] & \operatorname{im} d^2_{p,q} \arrow[r] \arrow[d, "\eta_2"] & 0 \\
0 \arrow[r] & \ker d^2_{p-1,q+2} \arrow[r] & \mathrm{E}^2_{p-1,q+2}(R_E;\mathbf{Z}/2) \arrow[r] & \operatorname{im} d^2_{p-1,q+2} \arrow[r] & 0
\end{tikzcd}$$

shows that $\eta_2 \colon \ker d^2_{p,q} \to \ker d^2_{p-1,q+2}$ is surjective for $p=0$ and bijective for $p \leqslant -1$. The claim for $r=3$ follows from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d^2_{p+2,q-1} \arrow[r] \arrow[d, "\eta_2"] & \ker d^2_{p,q} \arrow[r] \arrow[d, "\eta_2"] & \mathrm{E}^3_{p,q}(R_E;\mathbf{Z}/2) \arrow[r] \arrow[d, "\eta_2"] & 0 \\
0 \arrow[r] & \operatorname{im} d^2_{p+1,q+1} \arrow[r] & \ker d^2_{p-1,q+2} \arrow[r] & \mathrm{E}^3_{p-1,q+2}(R_E;\mathbf{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

The claim for v_1^4 is clear from Lemma 7, and the arguments above. We can only conclude for that part of the second quadrant which is not hit by differentials from the line $q=0$. □

The harmless assumption $p+q \geqslant 0$ is needed in some of the following. We keep that assumption implicit whenever we refer to eight periodicity. By using the calculations in Sections 4.1 and 4.2, and Lemma 16 we will next determine the d^2-differentials in (12).

Lemma 17. *Let $q \geqslant 1$. For the d^2-differentials in* (12) *we have*

$$\mathrm{rk}_2 \ker d^2_{p,q} = \begin{cases} (s_E + t_E) - (s_F + t_F) + 1 & \text{for } p = 0 \text{ and } q \text{ even}, \\ t_F^+ - t_F + 1 & \text{for } p \leqslant -1 \text{ and } q \text{ even}, \\ r_1 + r_2 + s_F + t_F - 1 & \text{for } p = 0 \text{ and } q \text{ odd}, \\ r_1 + 2(s_F + t_F) - (s_E + t_E) - 2 & \text{for } p \leqslant -1 \text{ and } q \text{ odd}, \end{cases}$$

and

$$\mathrm{rk}_2 \operatorname{im} d^2_{p,q} = \begin{cases} 2s_F + t_F^+ + t_F - (s_E + t_E) - 1 & \text{for } p \leqslant 0 \text{ and } q \text{ even}, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The assertion about the d^2-differentials originating from a row where q is even follows from the LHS spectral sequence, see Lemma 12. The two spectral sequences may be compared as follows.

Let Ω be the two integers in the maximal algebraic 2-ramified extension of F. We write G_E (resp. G_F) for the Galois group of the extension $R_E \to \Omega$ (resp. $R_F \to \Omega$). Everything in the proof is done for a filtering inductive system of finite Galois extensions of R_E contained in Ω. We use the notation $\mathrm{H}M$ for an Eilenberg–Mac Lane G_F-spectrum where $\pi_0(\mathrm{H}M) = M$ and where all the other homotopy groups of $\mathrm{H}M$ are zero. For a spectrum X we write $X[i]$ for its ith Postnikov section. The comparison is based on the G_F-equivariant diagram

$$\Sigma^{2i}\mathrm{HK}_{2i}/2(\Omega) \to \mathrm{K}/2(\Omega)[2i] \leftarrow \mathrm{K}/2(\Omega).$$

Here we use the existence of a G_F-equivariant Postnikov system for $\mathrm{K}/2(\Omega)$, see Theorem 1.2 in [26] and also p. 79 in [7]. The reference for a profinite Galois group is [18]. By passing to G_E homotopy fixed points we get the diagram

$$\Sigma^{2i}\mathrm{HK}_{2i}/2(\Omega)^{hG_E} \to \mathrm{K}/2(\Omega)[2i]^{hG_E} \leftarrow \mathrm{K}/2(R_E)$$

due to étale descent for totally imaginary number fields from [37]. The mod 2 algebraic K-groups of Ω are given by the formula

$$\mathrm{K}_*/2(\Omega) = \begin{cases} \mathbf{Z}/2(i) & \text{if } * = 2i \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we find

$$\begin{aligned} \pi_q \Sigma^{2i}\mathrm{HK}_{2i}/2(\Omega)^{hG_E} &= \pi_q \Sigma^{2i}\mathrm{H}\mathbf{Z}/2(i)^{hG_E} \\ &= \mathrm{H}^{2i-q}_{\text{ét}}(R_E; \mathbf{Z}/2(i)). \end{aligned}$$

Note that $\pi_q \Sigma^{2i}\mathrm{HK}_{2i}/2(\Omega)^{hG_E}$ is trivial unless $q{=}2i$, $2i{-}1$, or $2i{-}2$. Likewise, there are isomorphisms

$$\pi_*\mathrm{K}/2(\Omega)[2i]^{hG_E} = \begin{cases} 0 & \text{if } * \geqslant 2i+1, \\ \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2(i)) & \text{if } * = 2i, \\ \mathrm{H}^1_{\text{ét}}(R_E; \mathbf{Z}/2(i)) & \text{if } * = 2i-1, \\ \mathrm{H}^2_{\text{ét}}(R_E; \mathbf{Z}/2(i)) \oplus & \\ \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2(i-1)) & \text{if } * = 2i-2. \end{cases}$$

Next, we pass to C_2 homotopy fixed points, and find the diagram

$$(\Sigma^{2i}\mathrm{H}\mathbf{Z}/2(i)^{hG_E})^{hC_2} \to (\mathrm{K}/2(\Omega)[2i]^{hG_E})^{hC_2} \leftarrow \mathrm{K}/2(R_E)^{hC_2}$$

which may be identified with

$$\Sigma^{2i}\mathrm{H}\mathbf{Z}/2(i)^{hG_F} \to \mathrm{K}/2(\Omega)[2i]^{hG_F} \leftarrow \mathrm{K}/2(R_E)^{hC_2}.$$

Appendix B [19] gives a detailed discussion of generalized Atiyah–Hirzebruch spectral sequences. By specialization of Theorem B.8 in loc. cit. we have that the homotopy fixed point spectral sequence for the extension $R_F \to R_E$ is isomorphic to the spectral sequence gotten from the exact couple:

$$\begin{tikzcd}
\mathrm{K}/2(R_E)[i]^{hC_2} \arrow[rr] & & \mathrm{K}/2(R_E)[i-1]^{hC_2} \arrow[dl] \\
& \Sigma^i\,\mathrm{H}\,\mathrm{K}_i/2(R_E)^{hC_2} \arrow[ul] &
\end{tikzcd}$$

In effect, there are maps of spectral sequences

$$\begin{tikzcd}
\mathrm{H}^{-p}(C_2, \mathrm{H}^{2i-q}_{\text{ét}}(R_E;\mathbf{Z}/2(i))) \arrow[r, Rightarrow] \arrow[d, "\epsilon_{p,q}"'] & \mathrm{H}^{2i-p-q}_{\text{ét}}(R_F;\mathbf{Z}/2(i)) \arrow[d] \\
\mathrm{H}^{-p}(C_2, \mathrm{K}_q/2(\Omega)[2i]^{hG_E}) \arrow[r, Rightarrow] & \pi_{p+q}(\mathrm{K}/2(\Omega)[2i]^{hG_F}) \\
\mathrm{H}^{-p}(C_2, \mathrm{K}_q/2(R_E)) \arrow[r, Rightarrow] \arrow[u, "\rho_{p,q}"] & \pi_{p+q}(\mathrm{K}/2(R_E)^{hC_2}) \arrow[u]
\end{tikzcd}$$

where the lower displayed spectral sequence is isomorphic to the homotopy fixed point spectral sequence from the comments above, and where the upper displayed spectral sequence is just a reindexing of LHS into a homological spectral sequence. The idea is now to compare the d^2-differentials in these two spectral sequences via the d^2-differentials in the spectral sequence squeezed between them. The first step is to identify the maps $\varepsilon_{p,q}$ and $\rho_{p,q}$ for $q = 2i$, $2i-1$, and $2i-2$. From the calculations above we find

$$\varepsilon_{p,q} \text{ is } \begin{cases} \text{an isomorphism} & \text{for } q = 2i, 2i-1, \\ \text{an injection} & \text{for } q = 2i-2, \end{cases}$$

and

$$\rho_{p,q} \text{ is } \begin{cases} \text{a surjection} & \text{for } q = 2i, \\ \text{an isomorphism} & \text{for } q = 2i-1, 2i-2. \end{cases}$$

These claims follow easily from the description of the groups on the E^2-page of (12). To compare the d^2-differentials we look at the diagram

$$\begin{tikzcd}
\mathrm{E}^2_{p,q} \arrow[r, "{''d^2_{p,q}}"] \arrow[d, "\epsilon_{p,q}"'] & \mathrm{E}^2_{p-2,q+1} \arrow[d, "\epsilon_{p-2,q+1}"] \\
\mathrm{E}^2_{p,q} \arrow[r, "{'d^2_{p,q}}"] & \mathrm{E}^2_{p-2,q+1} \\
\mathrm{E}^2_{p,q} \arrow[r, "d^2_{p,q}"] \arrow[u, "\rho_{p,q}"] & \mathrm{E}^2_{p-2,q+1} \arrow[u, "\rho_{p-2,q+1}"']
\end{tikzcd}$$

Let $i \geqslant 2$. Then we can identify $d^2_{p,2i-2}$ with $'d^2_{p,2i-2}$. Consider now the diagram:

$$\begin{tikzcd}
\mathrm{H}^{-p}(C_2, \mathrm{H}^2_{\text{ét}}(R_E; \mathbf{Z}/2(i))) \arrow[r, "{''d^2_{p,2i-2}}"] \arrow[d] & \mathrm{H}^{-p+2}(C_2, \mathrm{H}^1_{\text{ét}}(R_E; \mathbf{Z}/2(i))) \arrow[d, "\cong"'] \\
\mathrm{H}^{-p}(C_2, \mathrm{K}_{2i-2}/2(R_E)) \arrow[r, "{'d^2_{p,2i-2}}"] \arrow[d] & \mathrm{H}^{-p+2}(C_2, \mathrm{K}_{2i-1}/2(R_E)) \\
\mathrm{H}^{-p}(C_2, \mathrm{H}^0_{\text{ét}}(R_E; \mathbf{Z}/2(i))) &
\end{tikzcd}$$

The left vertical map is split, and the splitting is gotten from the natural inclusion $R_{\mathbf{Q}(\sqrt{-1})} \to R_E$. Lemma 11 tells us that $\mathrm{H}^{-p+2}(C_2, \mathrm{K}_{2i-1}/2(R_{\mathbf{Q}(\sqrt{-1})}))$ is the trivial group. So $\operatorname{im} d^2_{p,2i-2}$ gets identified with $\operatorname{im} {''d^2_{p,2i-2}}$, and Lemma 12 concludes the proof for q even.

The claim for q odd is that $d^2_{p,q}$ is zero for all p. Lemma 16 says that

$$\eta_2 \colon \mathrm{E}^2_{p,q} \to \mathrm{E}^2_{p-1,q+2}$$

is surjective for $p = 0$, and an isomorphism for $p \leqslant -1$. Proposition 15 says that all the classes in bidegree $(0, 8n+1)$ are permanent cycles, so $d^2_{0,8n+1}$ is zero. Hence multiplication by η_2, cp. the diagram

$$\begin{tikzcd}
\mathrm{E}^2_{0,8n+1} \arrow[r] \arrow[d, "\eta_2"'] & \mathrm{E}^2_{-2,8n+2} \arrow[d, "\eta_2"] \\
\mathrm{E}^2_{-1,8n+3} \arrow[r] & \mathrm{E}^2_{-3,8n+4}
\end{tikzcd}$$

shows that $d^2_{-1,8n+3}$ is zero. Further multiplications by η_2 show that $d^2_{p,-2p+8n+1}$ is zero for $p \leqslant -2$. Eight-periodicity shows that $d^2_{p-4k,-2p+8n+1}$ is zero for $p \leqslant 0$ and $k \geqslant 0$. The classes in bidegree $(0, 8n+3)$ are permanent cycles according to Proposition 15. By the same argument as above, we find that $d^2_{p,-2p+8n+3}$ and $d^2_{p-4k-3,-2p+8n+1}$ are both zero for $p \leqslant 0$ and $k \geqslant 0$.

The classes in bidegrees $(0, 8n + 5)$ and $(0, 8n + 7)$ are not necessarily permanent cycles, so we need a slightly different argument for them. The d^2-cycle y_2, see Lemma 7, gives the diagram

$$\begin{tikzcd} \mathrm{E}^2_{p,q} \arrow[r, "d^2_{p,q}"] \arrow[d, "y"'] & \mathrm{E}^2_{p-2,q+1} \arrow[d, "\cong", "y"'] \\ \mathrm{E}^2_{p-2,q} \arrow[r, "d^2_{p-1,q}"] & \mathrm{E}^2_{p-4,q+1} \end{tikzcd}$$

We have just seen that half of the d^2-differentials with source in bidegree (p, q) where q is odd are zero, and the previous diagram tells us that the remaining ones are zero too. □

Lemma 17 determines the groups on the E^3-page of (12).

Corollary 18. *Let $q \geqslant 1$. For the E^3-page of (12) we have*

$$\mathrm{rk}_2\, \mathrm{E}^3_{p,q} = \begin{cases} (s_E + t_E) - (s_F + t_F) + 1 & \text{for } p = 0 \text{ and } q \text{ even}, \\ t_F^+ - t_F + 1 & \text{for } p \leqslant -1 \text{ and } q \text{ even}, \\ r_1 + r_2 + s_F + t_F - 1 & \text{for } p = 0 \text{ and } q \text{ odd}, \\ r_1 + 2(s_F + t_F) - (s_E + t_E) - 2 & \text{for } p = -1 \text{ and } q \text{ odd}, \\ r_1 - t_F^+ + t_F - 1 & \text{for } p \leqslant -2 \text{ and } q \text{ odd}. \end{cases}$$

Multiplication by η_2 induces a bijection $\mathrm{E}^3_{-1,q}(R_E; \mathbf{Z}/2) \to \mathrm{E}^3_{-2,q+2}(R_E; \mathbf{Z}/2)$ for q even.

Lemma 19. *Let $n \geqslant 0$.*

(i) *All the classes in bidegrees $(0, 8n + 1)$, $(0, 8n + 2)$ and $(0, 8n + 3)$ on the E^3-page of (12) are permanent cycles. It follows that all the classes in bidegrees $(-1, 8n + 3)$, $(-1, 8n + 4)$, $(-1, 8n + 5)$, $(-2, 8n + 5)$, $(-2, 8n + 6)$ and $(-2, 8n + 7)$ are permanent cycles.*

(ii) *All the classes in bidegrees $(p, -2p + 8n + 1)$, $(p, -2p + 8n + 2)$ and $(p, -2p + 8n + 3)$ on the E^3-page of (12) are infinite cycles for $p \leqslant -3$. None of these classes are permanent cycles. Hence $d^3_{0,8n+5}$, $d^3_{0,8n+6}$ and $d^3_{0,8n+7}$ are surjections.*

It follows that $d^3_{-1,8n+8}$ is an isomorphism, $d^3_{-1,8n+7}$ and $d^3_{-1,8n+9}$ are surjections, while the differentials $d^3_{p-3-4k,-2p+8n+3}$, $d^3_{p-2-4k,-2p+8n+2}$ and $d^3_{p-2-4k,-2p+8n+3}$ are isomorphisms for all $p \leqslant 0$ and $k \geqslant 0$.

(iii) *We have $\mathrm{rk}_2 \ker d^3_{0,8n+4} = (s_E + t_E) - (s_F + t_F)$, and $\mathrm{rk}_2 \operatorname{im} d^3_{0,8n+4} = 1$. Hence the group of permanent cycles in bidegree $(0, 8n + 4)$ is an elementary Abelian two group of rank $(s_E + t_E) - (s_F + t_F)$. For all $p \leqslant -1$ we have $\mathrm{rk}_2 \operatorname{im} d^3_{p,-2p+8n+4} = 1$ and $\mathrm{rk}_2 \ker d^3_{p,-2p+8n+4} = t_F^+ - t_F$. Moreover, for all $p \leqslant 0$ and all $k \geqslant 0$ there are equalities $\mathrm{rk}_2 \operatorname{im} d^3_{p-3-4k,-2p+8n+2} = 1$ and $\mathrm{rk}_2 \ker d^3_{p-3-4k,-2p+8n+2} = t_F^+ - t_F$.*

(iv) *All the infinite cycles in bidegree $(-3, 8n + 10)$ are hit by $d^3_{0,8n+8}$, and the group of permanent cycles in bidegree $(0, 8n + 8)$ is an elementary Abelian two group of rank $(s_E + t_E) - (s_F + t_F^+) + 1$.*

It follows that $\mathrm{rk}_2 \operatorname{im} d^3_{p,-2p+8n+8} = t_F^+ - t_F$ *and* $\mathrm{rk}_2 \ker d^3_{p,-2p+8n+8} = 1$ *for all* $p \leqslant -1$. *Moreover, there are equalities* $\mathrm{rk}_2 \operatorname{im} d^3_{p-1-4k,-2p+8n+2} = t_F^+ - t_F$ *and* $\mathrm{rk}_2 \ker d^3_{p-1-4k,-2p+8n+2} = 1$ *for all* $p \leqslant 0$ *and all* $k \geqslant 0$.

Proof. We commence with (i). Proposition 15 and Corollary 18 show that

$$\mathrm{rk}_2 \operatorname{im}(\mathrm{K}_{8n+2}/2(R_F) \to \mathrm{K}_{8n+2}/2(R_E)) = \mathrm{rk}_2\, \mathrm{E}^3_{0,8n+2},$$

so all the classes in bidegree $(0, 8n+2)$ on the E^3-page of (12) are permanent cycles. The E^2-page of (12) tells us that all the classes in bidegrees $(0, 8n+1)$ and $(0, 8n+3)$ are permanent cycles, as noted before. Lemma 16 shows that all the classes in the mentioned bidegrees can be written as η_2 times a permanent cycle. Those classes are all infinite cycles from the Leibniz rule for the pairing, and whence permanent cycles for bidegree reasons. This proves (i).

For (ii), note that the argument for (i) shows that all the mentioned classes are infinite cycles. Our claim is that these can not be permanent cycles. All the classes in question can be written as η_2^3 times a permanent cycle, so it suffices to prove that

$$\eta_2^3 \colon \mathrm{K}_n/2(R_E)^{hC_2} \to \mathrm{K}_{n+3}/2(R_E)^{hC_2}$$

is zero. In fact something stronger is true; the integral class η^3 is divisible by eight in $\mathrm{K}_3(\mathbf{Z})$ from [22]. The claims concerning the differentials from the axis $\mathrm{E}^3_{0,q}$ are obvious for bidegree reasons, since no other differentials can possibly hit the mentioned infinite cycles that are not permanent. Multiplication by η_2 and eight periodicity imply the remaining claims.

For (iii) we know $\mathrm{rk}_2 \operatorname{im} d^3_{0,8n+4} \geqslant 1$ since the class v_1^{4n+2} in bidegree $(0, 8n+4)$ in the mod 2 version of (3.1.1) does not survive, and it hits the non-trivial class $\eta_1^3 v_1^{4n}$ in bidegree $(-3, 8n+6)$. Proposition 15 tells us that

$$\mathrm{rk}_2 \operatorname{im}(\mathrm{K}_{8n+4}/2(R_F) \to \mathrm{K}_{8n+4}/2(R_E)) = (s_E + t_E) - (s_F + t_F)$$

so $\mathrm{rk}_2 \ker d^3_{0,8n+4} \geqslant (s_E + t_E) - (s_F + t_F)$. Hence $\mathrm{rk}_2 \operatorname{im} d^3_{0,8n+4} = 1$, by Corollary 18. This verifies the two-rank calculation of $\ker d^3_{0,8n+4}$ and $\operatorname{im} d^3_{0,8n+4}$, and gives the two rank of the group of permanent cycles in bidegree $(0, 8n+4)$ too. From the diagram

$$\begin{tikzcd}
\mathrm{E}^3_{0,8n+4} \arrow[r, "d^3"] \arrow[d, "\eta_2^{-p}"'] & \mathrm{E}^3_{-3,8n+6} \arrow[d, "\eta_2^{-p}", "\cong"'] \\
\mathrm{E}^3_{p,-2p+8n+4} \arrow[r, "d^3"] & \mathrm{E}^3_{p-3,-2p+8n+6}
\end{tikzcd}$$

we can identify $\operatorname{im} d^3_{p,-2p+8n+4}$ with $\operatorname{im} d^3_{0,8n+4}$. Recall that $\operatorname{im} d^3_{0,8n+4}$ has order one from above, and from Corollary 18 we find

$$\begin{aligned}
\mathrm{rk}_2 \ker d^3_{p,-2p+8n+4} &= \mathrm{rk}_2\, \mathrm{E}^3_{p,-2p+8n+4} - \mathrm{rk}_2 \operatorname{im} d^3_{p,-2p+8n+4} \\
&= t_F^+ - t_F.
\end{aligned}$$

for all $p \leqslant -1$. The rest of the claims in (iii) follow from multiplication by η_2 and eight periodicity, and we are left with (iv).

A variant of Lemma 16 shows that $\eta_2^3 \colon \ker d^3_{0,8n+4} \to \ker d^3_{-3,8n+10}$ is a surjection. By the same argument as for (2); $\ker d^3_{-3,8n+10}$ consists of infinite cycles that are not permanent. Multiplication by η_2 and (3) shows that $\mathrm{rk}_2\,\mathrm{im}\,d^3_{-3,8n+10} = 1$. We get $\mathrm{rk}_2\,\mathrm{im}\,d^3_{0,8n+8} = t_F^+ - t_F$ from Corollary 18, and also

$$\begin{aligned}\mathrm{rk}_2\ker d^3_{0,8n+8} &= \mathrm{rk}_2\,\mathrm{E}^3_{0,8n+8} - \mathrm{rk}_2\,\mathrm{im}\,d^3_{0,8n+8}\\ &= (s_E + t_E) - (s_F + t_F) + 1 - (t_F^+ - t_F)\\ &= (s_E + t_E) - (s_F + t_F^+) + 1.\end{aligned}$$

Proposition 15 shows that $\mathrm{rk}_2\ker d^3_{0,8n+8}$ equals the two rank of the image of $\mathrm{K}_{8n+8}/2(R_F)$ in $\mathrm{K}_{8n+8}/2(R_E)$. All the classes in $\ker d^3_{0,8n+8}$ are therefore permanent cycles. The trick with η_2 and eight periodicity concludes the proof of (4). □

Corollary 20. *Let $n \geqslant 0$. The two rank of the groups on the E^4-page of (12) are given as follows*:

$$\mathrm{rk}_2\,\mathrm{E}^4_{0,8n+i} = \begin{cases} r_1 + r_2 + s_F + t_F - 1 & \text{for } i = 1,3,\\ (s_E + t_E) - (s_F + t_F) + 1 & \text{for } i = 2,\\ (s_E + t_E) - (s_F + t_F) & \text{for } i = 4,\\ r_2 + s_F + t_F^+ & \text{for } i = 5,7,\\ (s_E + t_E) - (s_F + t_F^+) & \text{for } i = 6,\\ (s_E + t_E) - (s_F + t_F^+) + 1 & \text{for } i = 8,\end{cases}$$

$$\mathrm{rk}_2\,\mathrm{E}^4_{-1,8n+i} = \begin{cases} 2s_F + t_F^+ + t_F - (s_E + t_E) - 1 & \text{for } i = 1,7,\\ 1 & \text{for } i = 2,\\ r_1 + 2(s_F + t_F) - (s_E + t_E) - 2 & \text{for } i = 3,5,\\ t_F^+ - t_F + 1 & \text{for } i = 4,\\ t_F^+ - t_F & \text{for } i = 6,\\ 0 & \text{for } i = 8,\end{cases}$$

and

$$\mathrm{rk}_2\,\mathrm{E}^4_{-2,8n+i} = \begin{cases} 0 & \text{for } i = 1,2,3,\\ 1 & \text{for } i = 4,\\ r_1 - t_F^+ + t_F - 1 & \text{for } i = 5,7,\\ t_F^+ - t_F + 1 & \text{for } i = 6,\\ t_F^+ - t_F & \text{for } i = 8.\end{cases}$$

All the other groups are zero, and the E^4-page equals the E^∞-page for bidegree reasons. All the classes on the axis $\mathrm{E}^\infty_{0,q}$ are in the image of $\mathrm{K}_q/2(R_F)$. We have that multiplication by η_2 induces *a surjection*

$$\mathrm{E}^4_{0,q}(R_E;\mathbf{Z}/2) \to \mathrm{E}^4_{-1,q+2}(R_E;\mathbf{Z}/2)$$

and a surjection

$$\mathrm{E}^4_{-1,q}(R_E;\mathbf{Z}/2) \to \mathrm{E}^4_{-2,q+2}(R_E;\mathbf{Z}/2)$$

which is a bijection for q even. In particular, there is the class number inequality

$$2s_F + t_F^+ + t_F \geqslant (s_E + t_E) + 1.$$

Proof. The two-rank formulas follow from Lemma 19. The assertion concerning the classes on the axis $\mathrm{E}^\infty_{0,q}$ for $q \geqslant 1$ is plain from Proposition 15. Likewise for η_2, cp. Lemma 16. □

Proof. (Strong Quillen–Lichtenbaum conjecture.) From Corollary 20 we read off the following eight periodic E^∞-page of (12) for $q \geqslant 1$:

$r_1 - t_F^+ + t_F - 1$	$2s_F + t_F^+ + t_F - (s_E + t_E) - 1$	$r_2 + s_F + t_F^+$
$t_F^+ - t_F + 1$	$t_F^+ - t_F$	$(s_E + t_E) - (s_F + t_F^+)$
$r_1 - t_F^+ + t_F - 1$	$r_1 + 2(s_F + t_F) - (s_E + t_E) - 2$	$r_2 + s_F + t_F^+$
1	$t_F^+ - t_F + 1$	$(s_E + t_E) - (s_F + t_F)$
0	$r_1 + 2(s_F + t_F) - (s_E + t_E) - 2$	$r_1 + r_2 + s_F + t_F - 1$
0	1	$(s_E + t_E) - (s_F + t_F) + 1$
0	$2s_F + t_F^+ + t_F - (s_E + t_E) - 1$	$r_1 + r_2 + s_F + t_F - 1$
$t_F^+ - t_F$	0	$(s_E + t_E) - (s_F + t_F^+) + 1$

The classes on the axis $\mathrm{E}^\infty_{-1,q}$ are all in the image of $\mathrm{K}_q/2(R_F)$ since they can be written as the product of the class η_2 in $\mathrm{K}_1/4(R_F)$ by a class on the axis $\mathrm{E}^\infty_{0,q}$, cp. Lemma 7 and Corollary 20. Similarly, all the classes on the axis $\mathrm{E}^\infty_{-2,q}$ can be written as the product of the class η_2 in $\mathrm{K}_1/4(R_F)$ by a class on the axis $\mathrm{E}^\infty_{-1,q}$. This proves that the canonical map

$$\mathrm{K}_*/2(R_F) \to \mathrm{K}_*/2(R_E)^{hC_2}$$

is surjective for $* \geqslant 1$. Injectivity follows from Theorem 7.11 in [36] where the two-adic valuation of the number of elements in $\mathrm{K}_*/2(R_F)$ is given. Those numbers agree with the ones we read off from the E^∞-page of (12). □

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